Magnetic field modeling of a dual-magnet configuration

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This paper presents the theoretical and experimental studies of a dual-magnet (DM) configuration that forms the electromagnetic circuit of a nanopositioning actuator. Motivation of this work arises when an accurate prediction of the magnetic field behavior within the DM configuration is required to achieve ultrahigh precision motion control. In the theoretical modeling, the DM configuration is decomposed into several regions where each region is treated as a boundary-value problem. A method, termed superposition of the boundary conditions, is used to obtain the field solution of an air gap that is influenced by two magnetic sources. Consequently, a two-dimensional (2D) analytical model that accurately predicts the magnetic field behavior of the DM configuration is presented. In the experimental investigations, the magnetic flux density measured from a DM configuration prototype is used to validate the accuracy of the 2D analytical model. These experimental data were also compared against the magnetic flux density collected from a conventional single-magnet configuration prototype. Such comparisons verify the claimed features of the DM configuration, i.e., providing 40% increase in the magnetic flux density and offering an evenly distributed magnetic field through the entire air gap of 11 mm. © 2007 American Institute of Physics.

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I. INTRODUCTION

Manipulation between nano- and mesoscales with large payload and high bandwidth has always been a technological gap in the field of ultrahigh precision manufacturing. Traditionally, most ultrahigh precision manipulators are driven using piezoelectric (PZT) actuators due to their large actuating force and ease of control.\(^1\)\(^-\)\(^7\) However, PZT actuators are unsuitable for a manipulator targeted to achieve millimeters range with resolutions in nanometers due to their limited strokes.\(^8\)\(^-\)\(^10\) Consequently, electromagnetic (EM) actuation becomes a promising solution due to its capability of providing displacements in millimeters with infinite positioning resolutions, high accelerations, and fast actuating speed. Currently, EM propulsion, magnetic levitation (Maglev), and Lorentz-force actuation are the three main types of EM technique for realizing ultrahigh precision manipulations. An EM propulsion is achieved by the attraction and repulsion of a ferromagnetic moving part using the EM field generated from a solenoid.\(^11\)\(^-\)\(^16\) This technique possesses inconsistent actuating forces and eddy-current hysteresis due to the electromagnetization of the ferromagnetic stators. Maglev has been the most popular approach for developing supportless and contactless multiple degree-of-freedom (DOF) manipulators.\(^17\)\(^-\)\(^23\) However, Maglev manipulators have unstable behaviors and require complex control algorithms and costly control systems.

Among these techniques, Lorentz-force actuation is a direct noncommutation drive, which provides a constant output force with infinite positioning resolution within the entire traveling range without any complex control algorithm and systems.\(^24\)\(^-\)\(^27\) However, this technique offers a small output force and poor force-to-size ratio\(^27\) because it requires a very small effective air gap as magnetic flux density varies with respect to the distance from the magnet-polarized surface. Hence, this technique is rarely exploited for driving high-precision manipulators. Recently, a one-DOF nanopositioning actuator based on the Lorentz-force actuation is developed and it achieves the following:\(^28\) (1) a positioning accuracy of ±10 nm, (2) a large traveling range of 3 mm, (3) a high force sensitivity of 60 N/Amp, and (4) a fast actuating speed of 100 mm/s. This actuator, termed flexure-based electromagnetic linear actuator (FELA),\(^29\)\(^,\)\(^30\) consists of an electromagnetic driving mechanism (EDM) [Fig. 1(a)] and flexural-supporting bearings [Fig. 1(b)].

A dual-magnet (DM) configuration is introduced in this FELA to inherit the benefits of a Lorentz-force actuation while achieving a large continuous output force with small

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FIG. 1. (Color online) (a) An EDM with (b) flexural shims to form a complete (c) FELA.
input current from a compact-sized EDM (Fig. 2). This DM configuration is formed by a pair of rare-earth (NdFeB) permanent magnet (PM) [Fig. 2(a)] with an effective air gap between them. Unlike a conventional magnetic configuration (i.e., single magnet), a DM configuration has the following advantages:

(a) Larger effective air gap. A DM configuration ensures an evenly distributed magnetic flux density within a large effective air gap. It also ensures that the magnitude of the magnetic flux density remains unchanged even at tens of millimeters away from the PM polarized surface.

(b) Better force-to-size ratio. A DM configuration offers higher magnetic flux density within the effective air gap as compared to a conventional magnetic configuration. This allows larger force generation from a more compact magnetic circuit design.

(c) Lower heat generation. Heat introduced by electric current within the coil is the main source of thermal expansion that affects the positioning accuracy of an actuator. A DM configuration achieves large continuous output force with smaller amount of electric current to lower heat generation.

Currently, FELA uses a diffraction-based optical linear encoder to realize the feedback positioning control scheme. This encoder, which maintains a constant optical wavelength during the measurement, is more suitable for a less-stringent controlled environment as compared to laser interferometers. However, the positioning accuracy of FELA is limited to ±10 nm as the encoder resolution can only reach 5 nm. Nevertheless, a direct current control scheme can be implemented on FELA to improve its positioning accuracy since a Lorentz-force actuation can achieve infinite positioning resolutions. Using the linear current-force behavior of Lorentz-force actuation and the linear force-displacement relationship of flexural bearings, this direct current control scheme eliminates the need of a position encoder that limits the positioning accuracy of a FELA.

Prior to implementation, it is essential to establish an accurate current-force model of the EDM. In theory, the magnitude of Lorentz force \( F \) is governed by the magnetic flux density \( B \), the input current \( i \), and the orientation of the magnetic field and the current vector. With the coil length \( L \) being placed in the magnetic field, the Lorentz force is determined by

\[
F = i \int dL \times B. \quad (1)
\]

It is seen from Eq. (1) that the current-force relationship is affected by the coil length and magnetic flux density within the effective air gap. In this analysis, the coil length is a fixed constant and the magnetic flux density is assumed to vary within the effective air gap. Hence, the force along the \( x \) axis (or actuating direction), which will be generated by the current flowing within the coil along the \( z \) axis and magnetic flux density along the \( y \) axis, is expressed as

\[
F = i \int_{0}^{-\text{Coil}_z} (-\hat{z}dz) \times (B\hat{y})\hat{x} = iN\text{Coil}_z, \quad (2)
\]

where \( N \) is the number of coil turn and \( \text{Coil}_z \) is the coil length of each turn along the \( z \) axis.

Equation (2) indicates that an accurate current-force model requires a good prediction of the magnetic flux density in two-dimensional (2D) space. Precise prediction of the magnetic field gives a fundamental understanding of the magnetic field strength \( H \) and magnetic flux distribution in the design optimization stage of a DM configuration. These parameters also contribute to the complexity in control implementation to achieve the required performance specifications.

Initially, the study of magnetic flux density is theoretically conducted in one dimension (1D) using the magnetic charge principle.\(^3\) In the 1D theoretical analysis, Eq. (C3) (refer to Appendix) is proven to be an accurate model in predicting the magnetic flux density of a DM configuration based on experimental verifications. Unfortunately, the same principle becomes less effective in describing the magnetic field within the effective air gap of a DM configuration in higher dimensions. In this paper, a 2D analytical model that accurately predicts the magnetic flux density of a DM configuration is presented. Unlike previous efforts in solving boundary-value problems,\(^31–33\) superposition of boundary conditions is proposed to obtain the closed-form solution of the magnetic field in the effective air gap between two magnetic sources. Experimental data obtained from a DM configuration prototype are used to validate the accuracy of the proposed 2D analytical model. These data are also compared with measurements obtained from a conventional magnetic configuration.

II. MAGNETIC FIELD MODELING

The stator of an EDM is designed in a rectangular form with rectangular shaped NdFeB PMs [Fig. 3(a)]. Assuming that these rectangular PMs are uniformly magnetized, the magnetic field along the \( z \) axis will be similar to that along the \( x \) axis (Fig. 3). Hence, a three-dimensional (3D) problem can be reduced to a 2D problem. With its symmetrical design, a DM configuration can be further simplified into a 2D geometry that comprises of five regions (Fig. 3). Region I: half of the effective air gap of a DM configuration; region II: air breach between the PM-1 and the stator; region III: Half
of the PM-1 of a DM configuration; region IV: air breach between the PM-2 and the stator; and region V: half of the PM-2 of a DM configuration.

A. Assumptions
In this analysis, PMs are assumed to be ideal with field relationship described by the linear second quadrant of a PM demagnetization curve. The constitutive relation for a PM is expressed as

\[ \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}), \]

where \( \mu_0 \) is the permeability of the free space and \( \mathbf{M} \) is the magnetization of a PM.

The air gap is assumed to be a linear homogeneous media with the absence of magnetization, which leads to a constitutive relation for the air gap. This gives

\[ \mathbf{B} = \mu_0 \mathbf{H}. \]

With the assumption of a current-free air gap, the vector field becomes an irrotational field and the magnetic field strength is described by the divergence of the scalar potential \( \Phi \).

\[ \mathbf{H} = -\nabla \Phi. \]

B. Boundary-value problem
The magnetic field within a current-free environment is commonly described by Laplace’s equation, which is expressed as

\[ \nabla^2 \Phi = 0. \]

In this analysis, the method of separation of variables is used to obtain the solution of the scalar potential since Eq. (6) is a linear homogeneous partial differentiation equation. By letting the scalar potential to be a product of single variable function, this yields

\[ \Phi_i(x,y) = X_i(x)Y_i(y), \]

where \( i = 1,2,3,4,5 \) representing region I to region V, respectively (Fig. 3). Based on Eq. (7), Eq. (6) is further expanded into two independent terms and yields

\[ \frac{\ddot{X}_i(x)}{X_i(x)} + \frac{\ddot{Y}_i(y)}{Y_i(y)} = 0, \]

where each term is a function of a single variable that can be represented by an arbitrary constant, i.e., \( k_{ix}^2 + k_{iy}^2 = 0 \) with \( \dot{X}_i(x) = \frac{\partial X_i(x)}{\partial x} \) and \( \dot{Y}_i(y) = \frac{\partial Y_i(y)}{\partial y}. \)

In this analysis, \( k_{ix}^2 \) is chosen to be a negative value because with the magnetic field behavior is unpredictable along the \( x \) axis. On the other hand, \( k_{iy}^2 \) is chosen to be a positive value since the magnetic field increases or decreases according to the distance from the PM polarized surface along the \( y \) axis. Based on these conditions, Eq. (8) is separated into two independent ordinary differentiation equations (ODEs). Solving those ODEs yield

\[ X_i^n(x) = S^n_{xi} \cos(k^n_i x) + S^n_{xi} \sin(k^n_i x) \]

and

\[ Y_i^n(y) = S^n_{yi} e^{k^n_i y} + S^n_{yi} e^{-k^n_i y}, \]

where \( S^n_{xi}, j = 1,2,3,4 \) are constants.

Using Eqs. (7), (9), (10), and the superposition principle yields the general form of scalar potential solution for each region, which is expressed as

\[ \Phi_i(x,y) = \sum_{n=1}^{\infty} \left[ S^n_{xi} \cos(k^n_i x) + S^n_{xi} \sin(k^n_i x) \right] \times (S^n_{yi} e^{k^n_i y} + S^n_{yi} e^{-k^n_i y}), \]

where each constant is determined by imposing appropriate boundary conditions to each region.

C. Boundary conditions
In this analysis, superposition of boundary conditions is employed to obtain a closed-form solution for the scalar potential of the effective air gap between two magnetic sources, PM-1 and PM-2. The scalar potential of the effective air gap will be solved separately under the influence of each PM and subsequently superimposed together. Thus, the total scalar potential \( \Phi_{I,Total}(x,y) \) of the effective air gap is expressed as

\[ \Phi_{I,Total}(x,y) = \Phi_{I,PM-1}(x,y) + \Phi_{I,PM-2}(x,y), \]

where \( \Phi_{I,PM-1}(x,y) \) is the scalar potential of effective air gap under the influence of PM-1 and \( \Phi_{I,PM-2}(x,y) \) is that under the influence of PM-2.

Generally, initial boundary conditions are formulated under the following assumptions: (1) Permeability of the iron stator is infinite (\( \mu = \infty \)). (2) When analyzing the effective air gap under the influence of PM-1, PM-2 is treated as a free-air region. Thus, the tangential component vanishes when approaching \( c \) along \( y \) axis (refer to Fig. 3). (3) When analyzing the effective air gap under the influence of PM-2, PM-1 is treated as a free-air region. Thus, the tangential component vanishes when approaching 0 along \( y \) axis (refer to Fig. 3). (4) The normal component of the magnetic flux density in the middle of the effective air gap and the PMs is equal to zero (refer to Fig. 3 at \( x = b \)). (5) The tangential component of the magnetic field strength along the closed-loop path is equal to zero. (6) The orientation of the magnetization within a PM...
will only be perpendicular to the polarization surface. Thus, the magnetization of each region is defined by

$$M = \begin{cases} M_\text{III} & \text{region III, V} \\ 0 & \text{region I, II, IV} \end{cases} \quad (13)$$

**D. Air gap influence under PM-1**

**1. Scalar potential of region I (effective air gap)**

Based on the initial boundary condition stated in Sec. II C, postulation no. 5, we obtain

$$H_{\text{I,PM-1}}^n(0,y) = 0 \quad (a \leq y \leq b). \quad (14)$$

Applying Eq. (14) to Eq. (11) yields

$$\Phi_{\text{I,PM-1}}^n(x,y) = (C_{1,1} e^{2i\nu y} + C_{2,1} e^{-2i\nu y}) \sin(k_1^x x), \quad (15)$$

where $C_{1,1} = S_{1,1}^n$ and $C_{2,1} = S_{2,1}^n$. Based on the initial boundary condition stated in Sec. II C, postulation no. 4, we have

$$B_{\text{I,PM-1}}^n(l,y) = 0 \quad (a \leq y \leq b). \quad (16)$$

Applying Eq. (16) to Eq. (15) yields

$$k_1^x = \frac{(2n - 1) \pi}{2l} \quad n = 1, 2, 3, \ldots. \quad (17)$$

Based on the initial boundary condition stated in Sec. II C, postulation no. 2, we have

$$H_{\text{II,PM-1}}^n(x,c) = 0 \quad (g \leq x \leq l). \quad (18)$$

Applying Eq. (18) to Eq. (15) yields

$$C_{2,1} = -C_{1,1} e^{2i\nu y}. \quad (19)$$

Substituting Eqs. (17) and (19) into Eq. (15), along with the superposition principle, the scalar potential of region I under the influence of PM-1 is reexpressed as

$$\Phi_{\text{I,PM-1}}^n(x,y) = \sum_{n=1}^{\infty} C_{\text{I,PM-1}}^n \sin \left[ \frac{(2n - 1) \pi}{2l} x \right] \sinh \left[ \frac{(2n - 1) \pi}{2l} (y - c) \right], \quad (20)$$

where $C_{\text{I,PM-1}}^n$ is a constant.

**2. Scalar potential of region II (air breach 1)**

Based on the initial boundary condition stated in Sec. II C, postulation no. 5, we have

$$H_{\text{II}}^n(x,0) = 0 \quad (0 \leq x \leq g) \quad (21)$$

and

$$H_{\text{II}}^n(0,y) = 0 \quad (0 \leq y \leq a). \quad (22)$$

Applying Eq. (21) to Eq. (11) yields

$$\Phi_{\text{II}}^n(x,y) = [C_{1,1} \cos(k_2^x x) + C_{2,1} \sin(k_2^x x)] \sin(k_2^y y), \quad (23)$$

where $C_{1,1} = 2S_{3,1}^n y_{1,1}^n$ and $C_{2,1} = 2S_{2,1}^n y_{2,1}^n$ are constants.

Applying Eq. (22) to Eq. (23) yields

$$\Phi_{\text{II}}^n(0,y) = 0 \quad (0 \leq y \leq a). \quad (24)$$

Consequently, substituting Eq. (24) into Eq. (23) and applying the superposition principle reexpresses the scalar potential of region II, we have

$$\Phi_{\text{II}}^n(x,y) = \sum_{n=1}^{\infty} C_{\text{II}}^n \sin(k_2^x x) \sinh(k_2^y y), \quad (25)$$

where $C_{\text{II}}^n$ is a constant and $k_2^y$ is to be determined in the following section.

**3. Scalar potential of region III (PM-1)**

Based on the initial boundary condition stated in Sec. II C, postulation no. 5, we have

$$H_{\text{III}}^n(x,0) = 0 \quad (g \leq x \leq l). \quad (26)$$

Applying Eq. (26) to Eq. (11) yields

$$\Phi_{\text{III}}^n(x,y) = [C_{1,1} \cos(k_3^x x) + C_{2,1} \sin(k_3^x x)] \sinh(k_3^y y), \quad (27)$$

where $C_{1,1} = 2S_{3,1}^n y_{1,1}^n$ and $C_{2,1} = 2S_{2,1}^n y_{2,1}^n$ are constants.

The tangential component of magnetic field strength at the boundary between air breach 1 and PM-1 must be continuous. This condition gives

$$H_{\text{III}}^n(g,y) = H_{\text{III}}^n(g,y) \quad (0 \leq y \leq a). \quad (28)$$

Substituting Eqs. (23) and (27) into Eq. (28) yields

$$k_2^y = k_3^y, \quad (29)$$

$$C_{1,1}^n = C_{1,1}^n. \quad (30)$$

Substituting Eqs. (24) and (30) into Eq. (27) yields

$$\Phi_{\text{III}}^n(x,y) = C_{\text{Magnet}}^n \sin(k_3^x x) \sinh(k_3^y y), \quad (31)$$

where $C_{\text{Magnet}} = C_{2,1}^n$ is a constant.

Based on the initial boundary condition stated in Sec. II C, postulation no. 4, we have

$$B_{\text{III}}^n(l,y) = 0 \quad (0 \leq y \leq a). \quad (32)$$

Applying Eq. (32) to Eq. (31) yields

$$k_3^y = \frac{(2n - 1) \pi}{2l} \quad n = 1, 2, 3, \ldots. \quad (33)$$

Using superposition principle, the scalar potential of region III is expressed as

$$\Phi_{\text{III}}^n(x,y) = \sum_{n=1}^{\infty} C_{\text{Magnet}}^n \sin \left[ \frac{(2n - 1) \pi}{2l} x \right] \sinh \left[ \frac{(2n - 1) \pi}{2l} y \right]. \quad (34)$$
4. Scalar potential of region I under influence of PM-1

The tangential component of the magnetic field strength at the boundary between the effective air gap and PM-1 must be continuous. This condition gives

\[ H_{1,PM-1}(x,a) = H_{III}(x,a) \quad (g \leq x \leq l). \]  

(35)

Substituting Eqs. (20) and (34) into Eq. (35) yields

\[ C_{\text{Magnet1}}^n = C_{\text{Air-gap,PM-1}}^n \sinh \left[ \frac{(2n-1)\pi}{2l} (a - c) \right]. \]  

(36)

The tangential component of the magnetic flux density at the boundary between the effective air gap and PM-1 must be continuous. This condition gives

\[ B_{1,PM-1}(x,a) = B_{III}(x,a) \quad (g \leq x \leq l). \]  

(37)

Substituting Eqs. (20) and (34) into Eq. (37) yields

\[ \sum_{n=1}^{\infty} C_{\text{Air-gap,PM-1}}^n \frac{(2n-1)\pi}{2l} \sin \left[ \frac{(2n-1)\pi}{2l} x \right] U_1 = M, \]  

(38)

where

\[ U_1 = \sinh \left[ \frac{(2n-1)\pi}{2l} (a - c) \right] \coth \left[ \frac{(2n-1)\pi}{2l} a \right] - \cosh \left[ \frac{(2n-1)\pi}{2l} (a - c) \right]. \]  

(39)

Multiplying both sides of Eq. (38) by \( \sin \left[ (2n-1)\pi/2l \right] \) and integrates with respect to \( y \) yields

\[ C_{\text{Air-gap,PM-1}}^n = \frac{8MI(1)^n}{U_1(2n-1)^2}. \]  

(40)

Lastly, substituting Eq. (40) into Eq. (20) forms the complete solution for the scalar potential of the effective air gap (region I) under influence of PM-1 that is expressed as

\[ \Phi_{1,PM-1}(x,y) = \frac{8MI}{\pi^2} \sum_{n=1,2,3,\ldots}^{\infty} \frac{1}{U_1(2n-1)^2} \sin \left[ \frac{(2n-1)\pi}{2l} x \right] \times \sinh \left[ \frac{(2n-1)\pi}{2l} (y - c) \right]. \]  

(41)

E. Air gap influence under PM-2

1. Scalar potential of region I (effective air gap)

Based on the initial boundary condition stated in Sec. II C, postulation no. 3, we have

\[ H_{1,PM-2}(x,0) = 0 \quad (g \leq x \leq l). \]  

(42)

Applying Eq. (42) to Eq. (15) yields

\[ C_{1,1}^n = -C_{2,1}^n. \]  

(43)

Substituting Eqs. (17) and (43) into Eq. (15), along with superposition principle, the scalar potential of region I under the influence of PM-2 is reexpressed as

\[ \Phi_{1,PM-2}(x,y) = \sum_{n=1}^{\infty} C_{\text{Air-gap,PM-2}}^n \sin \left[ \frac{(2n-1)\pi}{2l} x \right] \times \sinh \left[ \frac{(2n-1)\pi}{2l} y \right], \]  

(44)

where \( C_{\text{Air-gap,PM-2}}^n = 2C_{1,1}^n \) is a constant.

2. Scalar potential of region IV (air breach 2)

Based on the initial boundary condition stated in Sec. II C, postulation no. 5, we have

\[ H_{IV,\gamma}(x,c) = 0 \quad (0 \leq x \leq g) \]  

(45)

and

\[ H_{IV,\gamma}(0,y) = 0 \quad (b \leq y \leq c). \]  

(46)

Applying Eq. (45) to Eq. (11) yields

\[ \Phi_{IV}^n = [C_{1,IV}^n \cos(k_4^n x) + C_{2,IV}^n \sin(k_4^n x)] \sin[k_4^n (y - c)], \]  

(47)

where \( C_{1,IV}^n = 2S_{1,IV}^n \gamma S_{1,IV}^n \) and \( C_{2,IV}^n = 2S_{2,IV}^n \gamma S_{2,IV}^n \) are constants.

Applying Eq. (46) to Eq. (47) yields

\[ C_{1,IV} = 0. \]  

(48)

Consequently, substituting Eq. (48) into Eq. (47) and applying superposition principle to reexpressed the scalar potential of region IV, we have

\[ \Phi_{IV}(x,y) = \sum_{n=1}^{\infty} C_{\text{Air-breath2}}^n \sin(k_4^n x) \sinh[k_4^n (y - c)], \]  

(49)

where \( C_{\text{Air-breath2}}^n = C_{2,IV}^n \) is a constant and \( k_4^n \) is to be determined in the following section.

3. Scalar potential of region V (PM-2)

Based on the initial boundary condition stated in Sec. II C, postulation no. 5, we have

\[ H_{IV,\gamma}(x,c) = 0 \quad (g \leq x \leq l). \]  

(50)

Applying Eq. (50) to Eq. (11) yields

\[ \Phi_{IV}^n = [C_{1,IV}^n \cos(k_4^n x) + C_{2,IV}^n \sin(k_4^n x)] \sin[k_4^n (y - c)], \]  

(51)

where \( C_{1,IV}^n = 2S_{1,IV}^n \gamma S_{1,IV}^n \) and \( C_{2,IV}^n = 2S_{2,IV}^n \gamma S_{2,IV}^n \) are constants.

The tangential component of magnetic field strength at the boundary between air breach 2 and PM-2 must be continuous. This condition gives

\[ H_{IV,\gamma}(g,y) = H_{IV,\gamma}(g,y) \quad (b \leq y \leq c). \]  

(52)

Applying Eq. (52) to Eq. (51) yields

\[ C_{1,IV} = C_{2,IV}^n. \]  

(53)
\[ k_{\alpha}^n = k_{\beta}^n. \] (54)

Substituting Eq. (53) to Eq. (51) yields
\[ \Phi_{\xi} = C_{\text{Magnet2}}^{\alpha} \sin(k_{\alpha}^n x) \sinh[k_{\beta}^n (y - c)], \] (55)
where \( C_{\text{Magnet2}}^{\alpha} = C_{2,Y}^{\alpha} \) is a constant.

Based on the initial boundary condition stated in Sec. II C, postulation no. 4, we have
\[ B_{\xi}^0(l, y) = 0 \quad (0 \leq y \leq a). \] (56)

Applying Eq. (56) to Eq. (55) yields
\[ k_{\beta}^n = \frac{(2n - 1)\pi}{2l}, \quad n = 1, 2, 3, \ldots \] (57)

Using superposition principle, the scalar potential of region \( V \) is expressed as
\[
\Phi_{\xi}(x, y) = \sum_{n=1}^{\infty} \frac{C_{\text{Magnet2}}^{\alpha}}{U_{\Pi}} \left[ \frac{(2n - 1)\pi}{2l} \right] x \sinh \left[ \frac{(2n - 1)\pi}{2l} (y - c) \right].
\] (58)

4. Scalar potential of region I under influence of PM-2

The tangential component of magnetic field strength at the boundary between the effective air gap and PM-2 must be continuous. This condition gives
\[ H_{\xi,\text{PM-2}}(x, b) = H_{\xi,\text{PM-1}}(x, b) \quad (g \leq x \leq l). \] (59)

Substituting Eqs. (44) and (58) into Eq. (59) yields
\[ C_{\text{Magnet2}}^{\alpha} = C_{\text{Air-gap,PM-2}}^{\alpha} \sinh \left[ \frac{(2n - 1)\pi}{2l} (b - c) \right]. \] (60)

The tangential component of magnetic flux density at the boundary between the effective air gap and PM-2 must be continuous. This condition gives
\[ B_{\xi,\text{PM-2}}(x, b) = B_{\xi,\text{PM-1}}(x, b) \quad (g \leq x \leq l). \] (61)

Substituting Eqs. (44) and (58) into Eq. (61) yields
\[ \sum_{n=1}^{\infty} \frac{C_{\text{Air-gap,PM-2}}^{\alpha}}{U_{\Pi}} \left[ \frac{(2n - 1)\pi}{2l} \right] x \sinh \left[ \frac{(2n - 1)\pi}{2l} (y - c) \right] U_{\Pi} = M, \] (62)

where
\[ U_{\Pi} = \sinh \left[ \frac{(2n - 1)\pi}{2l} b \right] \coth \left[ \frac{(2n - 1)\pi}{2l} (b - c) \right] - \cosh \left[ \frac{(2n - 1)\pi}{2l} b \right]. \] (63)

Multiplying both sides of Eq. (62) by \( \sin[(2n - 1)\pi/2l] \) and integrating with respect to \( x \) yields
\[ \Phi_{\xi,\text{PM-2}}(x, y) = \frac{8Ml(1)^n}{U_{\Pi}(2n - 1)^2} \sum_{n=1, 2, 3, \ldots}^{\infty} \frac{1}{U_{\Pi}(2n - 1)^2} \sin \left[ \frac{(2n - 1)\pi}{2l} x \right] \sinh \left[ \frac{(2n - 1)\pi}{2l} (y - c) \right]. \] (65)

F. Proposed 2D analytical model

Based on Eqs. (4), (5), and (12), the magnetic flux density within the effective air gap of a DM configuration is expressed as
\[ B_{\xi,\text{Total}}(x, y) = -\mu_0 \nabla \left[ \Phi_{\text{LM-1}}(x, y) + \Phi_{\text{LM-2}}(x, y) \right]. \] (66)

By substituting Eqs. (41) and (65) into Eq. (66), the tangential component of the magnetic flux density within the effective air gap of a DM configuration is described by the following analytical model:
\[ B_{\xi,\text{Total}}(x, y) = -\frac{4\mu_0 M}{\pi} \sum_{n=1, 2, 3, \ldots}^{\infty} \frac{1}{(2n - 1)} \times \sin \left[ \frac{(2n - 1)\pi}{2l} x \right] \left\{ \frac{1}{U_{\Pi}} \cosh \left[ \frac{(2n - 1)\pi}{2l} (y - c) \right] + \frac{1}{U_{\Pi}} \sinh \left[ \frac{(2n - 1)\pi}{2l} (y - c) \right] \right\}. \] (67)

III. EXPERIMENT

A. Prototype

In this work, two types of magnetic circuit prototypes are developed for evaluation and comparison. One is the proposed DM configuration [Fig. 4(a)], and the other is a conventional magnetic configuration, using a single PM and a closed-loop path [Fig. 4(b)]. Both prototypes employ 54 \( \times \) 50 \( \times \) 7.5 mm\(^3\) NdFeB PMs (type N45M) with remanence magnetic flux density \( T \) of 1.33 Tesla and a maximum operating temperature of 120 \( ^\circ \)C. Both prototypes have an effective air gap of 11 mm.
B. Experimental setup

A LAKESHORE Hall-sensor probe and Gauss meter are two measurement tools employed in the experimental investigations. The Hall-sensor probe is a single-axis magnetic flux density measuring device attached to a three-axis translational manipulator, which is programmed to position the probe at an incremental step of 0.5 mm in the X, Y, and Z directions within the effective air gap. The Gauss meter measures the amount of magnetic flux density instantaneously with a noise resolution of ±0.001 T and records the data via a personal computer. Based on the assumption that the magnetic flux density distribution is symmetrical for both halves of the effective air gap volume, half of the air gap volume (from the middle to the edge of a PM) has been measured for both prototypes (Fig. 5). Due to the size of the Hall-sensor probe (~1.75 mm in diameter) the allowable height for measurement is restricted to 7 mm, i.e., between 2 and 9 mm of the effective air gap along the z axis (Fig. 5). Thus the measured volume within the air gap is $27 \times 50 \times 7$ mm$^3$ (Fig. 5).

IV. RESULTS

A. Magnitude of magnetization

To predict the magnetic flux density within the effective air gap of the DM configuration, the magnetization of the PM is required as shown in the proposed 2D analytical model [Eq. (67)]. In the literatures, the magnetization of the PM is usually assumed to be proportional to the remanence magnetic flux density of the PM, i.e., $B_r = \mu_0 M_r$ or to have the equivalent magnitude as the coercive force $H_c$. Some authors also suggested that the magnetization lies between a range from $H_c$ to $B_r$ for each particular PM. Nevertheless, none of the existing literatures have provided an analytical model that accurately determines the magnetization of the PM from these magnetic parameters. In this work, an effective empirical approach, which is well demonstrated by Lee, will be used to estimate the magnetization of a PM. Based on this approach, the magnetic flux density that emanates from the middle of a PM sample (used in the prototypes) is measured at 3.5, 7, and 10.5 mm along the y axis from the PM surface using the Gauss meter. The measured values are substituted into Eq. (C3) with $a=25$ mm, $b=27.5$ mm, and $L=7.5$ mm to estimate the actual magnetization of the PM sample. As a result, the calculated magnitude of the magnetization is obtained as $899.23 \times 10^3$ A/m.

B. Effect of iron separating the symmetrical DM configuration

An EDM consists of a symmetrical DM configuration within its stator (Fig. 2). At the center of the stator, both DM configurations are separated by an 8 mm thick iron, which forms a closed-loop path for the magnetic flux flow. Unfortunately, the experiments conducted in this work have shown otherwise based on the developed prototype (Fig. 6). Using the conventional magnetic configuration prototype, the magnetic flux density within the effective air gap that emanates from PM-A is measured [Fig. 6(a)]. Next, PM-B is added beneath PM-A, where both PMs face each other with the same pole and are separated by an 8 mm thick iron path. Subsequently, the magnetic flux density within the effective air gap is remeasured [Fig. 6(b)]. A comparison between both measurements shows a drop of the magnetic flux density that emanates from PM-A after PM-B is added. It shows that the thickness of the iron is ineffective in guiding the magnetic flux back to each respective DM configuration. Such ineffectiveness, which leads to the drop of magnetic flux density within the effective air gap, needs to be considered in the prediction of the magnetic field behavior using the proposed analytical model. Consequently, difference between both measurements are used to back calculate the magnetization using Eqs. (4), (5), and (41) with $a=7.5 \times 10^{-3}$ m, $c=18.5 \times 10^{-3}$ m, $l=32 \times 10^{-3}$ m, and $\mu_0=4 \pi \times 10^{-7}$, which are the parameters obtained from the actual prototype. This approach returns a magnetization of $103.45 \times 10^3$ A/m, indicating loss of magnetic flux density due to the ineffectiveness of separating iron. It is assumed that the calculated magnetization will be doubled in magnitude due to a DM configuration.
C. DM versus conventional magnetic configuration

Magnetic flux densities within the effective air gap of the conventional magnetic configuration and the DM configuration are measured experimentally along the X-Y plane at Z = 25 mm (refer to Fig. 5, at the z axis), and are plotted in Fig. 7. In the DM configuration, the magnetic flux density within the effective air gap is 40% higher [Fig. 7(a)] as compared to a conventional magnetic configuration [Fig. 7(b)]. Based on Fig. 7, the magnetic flux leakage only occurs between 0 and 5 mm of the effective air gap within the DM configuration, while such leakages occur between 0 and 15 mm of the effective air gap within the conventional magnetic configuration. This shows that the magnetic flux density is uniformly distributed between 5 and 27 mm of the effective air gap within the DM configuration. In addition, the magnetic flux density within the effective air gap of a conventional magnetic configuration reduces as it moves away from the PM face, while magnetic flux density within the effective air gap of the DM configuration remains unchanged.

Figure 8 plots the magnetic flux density within the effective air gap of a conventional magnetic configuration and a DM configuration, which is experimentally measured along the X-Z plane at Y = 2 mm and Y = 7 mm (refer to Fig. 5, at y axis). For a conventional magnetic configuration, Fig. 8(a)
shows that the magnetic flux density of 0.35 T is registered evenly across the effective air gap at $Y=2$ mm. Yet at $Y=7$ mm, the same amount of magnetic flux density can only be recorded near the middle portion across the effective air gap. On the other hand, Fig. 8 shows that the magnetic flux density of 0.52 T is registered evenly across the effective air gap at $Y=2$ mm and $Y=7$ mm. This shows that a DM configuration offers a constant distribution of magnetic flux density across the effective air gap regardless of the distance from the PM surface. These experimental results have shown that a DM configuration is able to maintain an evenly distributed magnetic field throughout a large effective air gap, which cannot be achieved using the conventional magnetic configuration.

D. Analytical results versus experimental data

In Fig. 9, the magnetic flux density measured experimentally along the X-Y plane at $Z=25$ mm (refer to Fig. 5, at $z$ axis) is plotted [Fig. 9(a)] against the results obtained from the proposed 2D analytical model [Eq. (67)] [Fig. 9(b)] and the numerical analysis [Fig. 9(c)], respectively (refer to Appendix A for the parameters used). A comparison between Figs. 9(a) and 9(b) shows that the 2D analytical model has made an accurate prediction on the magnitude of the magnetic flux density and the magnetic field behavior between 5 and 27 mm of the 2D plane of the effective air gap. Magnetic flux leakage between 0 and 5 mm of the 2D plane of the effective air gap is also well predicted by the 2D analytical model. Comparatively, Fig. 9(c) shows that the numerical analysis predicts a slightly lower magnetic flux density, i.e., ~0.46 T, between 10 and 27 mm of the 2D plane of the...
effective air gap. A larger area of magnetic flux leakage derived from the numerical field solution also suggests that the 2D analytical model offers a better prediction on the magnetic flux distribution for a DM configuration.

The difference in the magnetic flux density between the analytical and the experimental results is plotted in Fig. 10. It shows that the 2D analytical model accurately predicts the magnitude of the magnetic flux density throughout the 2D plane of the effective air gap with a deviation of ±0.02 T. Based on Fig. 10, a slight inaccuracy is found at the two corners of the effective air gap, i.e., from 0 to 1 mm. Such inaccuracies are accountable for as the actual PMs used in the prototypes have fillet edges instead of sharp 90° edges, which are assumed during the formulation of the 2D analytical model (refer to Fig. 4). As a result, more magnetic flux leakage is registered as projected at the two corner edges of the effective air gap in Fig. 9(a).

The magnetic flux density measured experimentally along the Y-Z plane at X=27 mm (refer to Fig. 5, at x axis) is plotted in Fig. 11(a). For comparison, the same analytical result obtained from the 2D analytical model is plotted in Fig. 11(b). It has shown that the 2D analytical model is also accurate in predicting the magnitude of the magnetic flux density along the Y-Z plane. Figure 12 plots the difference in the magnetic flux density between the analytical and the experimental results. It shows that the 2D analytical model can accurately predict the magnitude of the magnetic flux density throughout the 2D plane of the effective air gap with a deviation of ±0.02 T. Similarly, slight inaccurate predictions occur at the two corners of the effective air gap region due to

![Image](image-url)

**FIG. 10.** (Color online) Difference in magnetic flux density between the results obtained from the proposed analytical model and the experiments (X-Y plane at Z=25 mm).

![Image](image-url)

**FIG. 11.** (Color online) Magnetic flux distribution in Y-Z plane at X =27 mm: (a) experimental data and (b) results from the proposed analytical model.
the manufacturing defects of the actual PMs used in the prototypes. Nevertheless, all these results have shown the effectiveness of the 2D analytical model in predicting the magnetic field behavior within the effective air gap of a DM configuration in both X-Y and Y-Z planes.

V. CONCLUSIONS

This paper describes a 2D analytical model that predicts the magnetic field behavior within the effective air gap of a DM configuration in 2D space. The magnetic field within the effective air gap is analytically represented in the scalar potential form and solved as a boundary-value problem. The closed-form solution of the scalar potential is used for predicting the magnetic field behavior within the effective air gap of a DM configuration. A comparison between the analytical and experimental results have shown that this 2D analytical model offers an accurate prediction of the magnitude of the magnetic flux density across the 2D plane of the effective air gap with a deviation of ±0.02 T. Hence, it is useful for rapid evaluation and parametric design of a DM configuration. Most importantly, an accurate current-force model of an EDM can be established based on this 2D analytical model. With a current-force model and a force-displacement model of the flexural bearings, a direct-current control can be implemented on a FELA so as to avoid the limitations of encoders and to achieve a high positioning accuracy. A conventional magnetic configuration is also built to validate the effectiveness of a DM configuration. Experimental data have shown that a DM configuration can provide 40% increase in the magnetic flux density within the effective air gap. It also shows that a DM configuration offers an evenly distributed magnetic flux density through the entire air gap of 11 mm. These results support the use of a DM configuration in aiding FELA to achieve the nanopositioning and large thrust force requirements.

APPENDIX A: PARAMETERS USED IN THE PROPOSED 2D ANALYTICAL MODEL FOR MAGNETIC FIELD MODELLING

The parameters required for obtaining a magnetic field behavior from the proposed 2D analytical model are as follows:

1. $M = 692.33 \times 10^3 \quad (\text{A/m})$ [Note: Obtained from $899.23 \times 10^3 - (103.45 \times 10^3)2 \text{ PMs}$]
2. $\mu_0 = 4\pi \times 10^{-7} \quad (\text{Wb/Am})$
3. $a = 7.5 \times 10^{-3} \quad (\text{m})$
4. $b = 18.5 \times 10^{-3} \quad (\text{m})$
5. $c = 26 \times 10^{-3} \quad (\text{m})$
6. $g = 3 \times 10^{-3} \quad (\text{m})$
7. $l = 30 \times 10^{-3} \quad (\text{m})$

APPENDIX B: PARAMETERS USED IN THE FINITE ELEMENT ANALYSIS

In this work, ANSYS 10 was used to conduct finite element analyses on a 2D model of a DM configuration. The parameters used to obtain the field solution within the effective air gap of a DM configuration (refer to Fig. 3, Region 1) are as follows:

1. Element: PLANE 53, eight nodes per element
2. air regions: relative permeability, $\mu_r = 1$
3. ndFeB PMs: coercive force $H_c = 692.33 \times 10^3 \quad (\text{A/m})$
4. iron: relative permeability $\mu_r = 200$
5. permeability in space: $\mu_0 = 4\pi \times 10^{-7} \quad (\text{Wb/Am})$
6. boundary of the model: vector potential and flux parallel on line
7. solver: static analysis and Magnetic Vector Potential (MVP) formulation.

The 2D model was dimensioned according to the DM configuration prototype. For the effective air gap, each element, which is formed by eight nodes, is sized as a 0.5 mm square.
The magnetic flux density $B_y$ of each element is obtained by averaging the values of all the nodes that form the element.

**APPENDIX C: A CONVENTIONAL MAGNETIC FIELD MODEL**

Currently the magnetic field of a PM is commonly analyzed using an analytical model that describes a PM with magnetic charge. Based on this principle, Furlani\(^{34}\) made a comprehensive derivation of this model, and the scalar potential that is used to describe the magnetic field of the above PM can be expressed as

$$
\Phi = -\frac{1}{4\pi} \int \int \frac{\nabla' \cdot \mathbf{M}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} dV' + \frac{1}{4\pi} \oint_S \frac{\mathbf{M}(\mathbf{x}) \cdot \mathbf{n}}{|\mathbf{x} - \mathbf{x}'|} ds',
$$

(C1)

where the solution gives a sum of the magnetic charge volume $V$ and the surface $S$ confining it. In addition, $\mathbf{x}$ is the vector of the observation point, $\mathbf{x}'$ is the vector of source point, $\nabla'$ operates on the primed coordinates, and $\mathbf{n}$ is the outward unit normal to the surface.

For 1D analysis of the magnetic flux density above a PM, it is assumed that the magnetization $\mathbf{M} = M_0 \mathbf{y} - \mathbf{M} = 0$. Using Eqs. (4) and (C1), $\mathbf{x} = y' \hat{y}$ and $\mathbf{x}' = x' \hat{x} + z' \hat{z}$, the magnetic flux density along the $y$ axis for a DM configuration is expressed as

$$
B_y^{\text{Total}}(d) = B_y^{\text{PM-1}}(d) + B_y^{\text{PM-2}}(d - g),
$$

(C2)

where

$$
B_y(\zeta) = \frac{\mu_0 M}{\pi} \left\{ \tan^{-1}\left[ \frac{(\zeta + L) \sqrt{a^2 + b^2 + (\zeta + L)^2}}{ab} \right] 
- \tan^{-1}\left[ \frac{p \sqrt{a + b + \zeta}}{ab} \right] \right\},
$$

(C3)

with $a =$ half the width of magnet, $b =$ half the length of magnet, $L =$ thickness of magnet, and $\zeta$ represents the distance from the surface of PM-1, i.e., $d$, or from the surface PM-2, i.e., $d - g$ (where $g =$ height of air gap).


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\(^{10}\)P. J. Falter, Diamond Turning of Nonrotationally Symmetric Surfaces (North Carolina State University, Raleigh, NC, 1990).


